ASYMPTOTIC EVOLUTION OF SMOOTH CURVES UNDER GEODESIC FLOW ON HYPERBOLIC MANIFOLDS-II

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Abstract. Extending the earlier results for analytic curve segments, in this article we describe the asymptotic behaviour of evolution of a finite segment of a C^n-smooth curve under the geodesic flow on the unit tangent bundle of a finite volume hyperbolic n-manifold. In particular, we show that if the curve satisfies certain natural geometric conditions, the pushforward of the parameter measure on the curve under the geodesic flow converges to the normalized canonical Riemannian measure on the tangent bundle in the limit. We also study the limits of geodesic evolution of shrinking segments.

We use Ratner's classification of ergodic invariant measures for unipotent flows on homogeneous spaces of $\mathrm{SO}(n,1)$, and an observation relating local growth properties of smooth curves and dynamics of linear $\mathrm{SL}(2,\mathbb{R})$-actions.

1. Introduction

Let M be a hyperbolic n-dimensional manifold of finite volume, $p : T^1(M) \to M$ be the unit tangent bundle over M, and $\{g_t\}_{t\in\mathbb{R}}$ denote the geodesic flow on $T^1(M)$. Let $\pi : \mathbb{H}^n \to M$ be a locally isometric universal cover of M and $D\pi : T^1(\mathbb{H}^n) \to T^1(M)$ the corresponding covering map. If $\{\tilde{g}_t\}$ denotes the geodesic flow on $T^1(\mathbb{H}^n)$, then $p(\tilde{g}_t(v)) \xrightarrow{t\to\infty} \mathrm{Vis}(v)$ for all $v \in T^1(\mathbb{H}^n)$, where $\mathrm{Vis} : T^1(\mathbb{H}^n) \to \partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$ denotes the visual map. We define

$$\mathscr{S} = \{\partial\mathbb{H}^m \subset \mathbb{S}^{n-1} : \mathbb{H}^m \hookrightarrow \mathbb{H}^n \text{ is an isometric embedding such that } \pi(\mathbb{H}^m) \text{ is closed in } M, \text{ where } 2 \le k \le n-1\}. \tag{1}$$

Then $\mathscr{S}$ is a countable collection of proper closed subspheres of $\mathbb{S}^{n-1}$ ([10],[11, §(5.2)]).

Let I be a compact interval with nonempty interior. Let $\psi : I \to T^1(M)$ be an continuous map with the following property: If $\tilde{\psi} : I \to T^1(\mathbb{H}^n)$ is any continuous lift of ψ under $D\pi$, then

a) $\mathrm{Vis}\circ\tilde{\psi} \in C^n(I, \mathbb{S}^{n-1})$,

b) the first derivative $(\mathrm{Vis}\circ\tilde{\psi})^{(1)}(s) \neq 0$ for Lebesgue a.e. $s \in I$, and

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c) for any $S \in \mathscr{S}$, $\mathrm{Vis}(\tilde{\psi}(s)) \notin S$ for Lebesgue a.e. $s \in I$.

Theorem 1.1. *Let the notation be as above. Then for any $f \in \mathrm{C_c}(T^1(M))$*

$$\frac{1}{|I|}\int_I f(g_t\psi(s))\,ds \xrightarrow{t\to\infty} \int_{T^1(M)} f\,d\mu, \tag{2}$$

where $|I|$ denotes the Lebesgue measure of I and μ denotes the normalized measure associated to the canonical Riemannian volume form on $T^1(M)$.

For the motivation for considering the question the reader is referred to [12], where the result was proved in the special case of analytic curve segments ψ. The proof in [12] involves the use of '(C,α)-growth properties', in the sense of Kleinbock and Margulis [5], of finite-dimensional spaces of analytic functions. As these could not be extended to smooth functions, the analogous result could not be proved by the techniques in [12] for smooth curve segments, though the conclusion could be expected to hold in that generality, as was especially commented to the author by Peter Sarnak in response to the result in [12].

In this article we overcome this difficulty by making a new observation of a linear dynamical nature. It implies that if we approximate an arbitrarily short piece of a C^n-curve by a polynomial curve degree at most n, then the geodesic flow expands both the approximating curves into long curves of roughly fixed lengths while still keeping them sufficiently close. This observation allows us to use the growth properties of polynomial curves of bounded degrees for linearization method [1, 11, 2, 6].

On the space $C^n(I, T^1(M))$, we consider the topology of uniform convergence up to n-derivatives. We now state a more robust form of Theorem 1.1.

Theorem 1.2. *Let the map ψ be as above. Then given $f \in \mathrm{C_c}(T^1(M))$ and $\epsilon > 0$ there exists a neighbourhood Ω of ψ in $C^n(I, T^1(M))$ and $T > 0$ such that*

$$\left|\frac{1}{|I|}\int_I f(g_t\psi_1(s))\,ds - \int_{T^1(M)} f\,d\mu\right| < \epsilon, \quad \forall \psi_1 \in \Omega,\ \forall t > T. \tag{3}$$

It may be noted that even for an analytic map ψ, the above uniform version could not be proved using the methods of [12], because we do not have the (C,α)-growth property for linear span a neighbourhood of an analytic function.

1.1. **Evolution of general C^n-curves.** Let $\bar{\mathscr{S}}$ denote the collection of all closed totally geodesic immersed submanifolds of M (including M itself). Given $M_1 \in \bar{\mathscr{S}}$, let $\mathscr{S}(M_1) \subset \mathscr{S} \cup \{\mathbb{S}^{n-1}\}$ be the collection of the boundaries of all possible lifts of M_1 in $\mathbb{H}^n$. Let μ_{M_1} denote the probability measure which is the normalized measure corresponding to the canonical Riemannian volume form on $T^1(M_1) \subset T^1(M)$. Given any $S \in \mathscr{S}$ or $S = \mathbb{S}^{n-1}$, define

$$S^* = S \setminus \bigcup_{\substack{S_1 \subset S,\ \dim S_1 < \dim S \\ S_1 \in \mathscr{S}}} S_1. \tag{4}$$

Let $\psi \in C^n(I, T^1(M))$. Let $\tilde{\psi} \in C^n(I, \mathbb{H}^n)$ denote a lift of ψ under $D\pi$. We define

$$I(S) = \{s \in I : \mathrm{Vis}(\tilde{\psi}(s)) \in S^*\}, \tag{5}$$

$$I(M_1) = \cup_{S \in \mathscr{S}(M_1)} I(S). \tag{6}$$

In particular,

$$I(M) = I(\mathbb{S}^{n-1}) = \{s \in I : \mathrm{Vis}(\tilde{\psi}(s)) \notin S,\ S \in \mathscr{S}\}.$$

Theorem 1.3. *Suppose that* $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(s) \neq 0$ *for almost all* $s \in I$*. Then given any* $f \in \mathrm{C_c}(T^1(M))$*,*

$$\lim_{t\to\infty} \int_I f(a_t\psi(s))\, ds = \sum_{M_1 \in \bar{\mathscr{S}}} |I(M_1)| \int_{T^1(M)} f\, d\mu_{M_1}. \tag{7}$$

The above statement is obtained as a consequence of results about limiting distributions of the evolution of shrinking curves under the geodesic flow (see §7).

1.2. **Flows on homogeneous spaces.** The above results will be derived from their analogues in terms of dynamics of flows on homogeneous space of Lie groups.

Let $G = \mathrm{SO}(n,1) = \mathrm{SO}(Q_n)$, where Q_n is a quadratic form in $(n+1)$ real variables, defined as

$$Q_n(y, x_1, \ldots, x_{n-1}, z) = 2yz - (x_1^2 + \cdots + x_{n-1}^2). \tag{8}$$

Let Γ a lattice in G. For $t \in \mathbb{R}$ and $\boldsymbol{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we define

$$a_t = \begin{bmatrix} e^t & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & e^{-t} \end{bmatrix} \in G \quad \text{and} \quad u(\boldsymbol{x}) = \begin{bmatrix} 1 & x_1 & \ldots & x_{n-1} & \|\boldsymbol{x}\|^2/2 \\ & 1 & & & x_1 \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1} \\ & & & & 1 \end{bmatrix} \in G. \tag{9}$$

Theorem 1.4. *Let* I *be a compact interval with nonempty interior and* $\varphi : I \to \mathbb{R}^{n-1}$ *be a* C^n*-map such that* $\varphi^{(1)}(s) \neq 0$ *for all* $s \in I$*, and for any sphere or a proper affine subspace* S *in* $\mathbb{R}^{n-1}$*,*

$$|\{s \in I : \varphi(s) \in S\}| = 0. \tag{10}$$

Let $\varphi_k \xrightarrow{k\to\infty} \varphi$ *be a convergent sequence in* $C^n(I, \mathbb{R}^{n-1})$*,* $x_k \to x_0$ *a convergent sequence in* G/Γ *and* $t_k \to \infty$ *in* $\mathbb{R}$*. Then for any* $f \in \mathrm{C_c}(G/\Gamma)$*,*

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f(a_{t_k} u(\varphi_k(s)) x_k)\, ds = \int_{G/\Gamma} f\, d\mu_G, \tag{11}$$

where μ_G *is the unique* G*-invariant probability measure on* G/Γ*.*

In fact, we shall obtain the following more general version. Let $P^- = \{g \in G : \overline{\{a_t g a_t^{-1} : t > 0\}}$ is compact$\}$. Then P^- is a proper parabolic subgroup of G and $P^- \backslash G$ naturally identifies with $\mathrm{SO}(n-1)\backslash \mathrm{SO}(n) \cong \mathbb{S}^{n-1}$. Let $\mathcal{I} : G \to P^- \backslash G \cong \mathbb{S}^{n-1}$ be the corresponding map. We note that under this

identification G acts on $\mathbb{S}^{n-1}$ by conformal transformations. For $m \geq 2$ and $g \in G$, $\mathcal{I}(\mathrm{SO}(m,1)g)$ is a subsphere of $\mathbb{S}^{n-1}$ of dimension $m-1$. We note that $\mathcal{I}(N_G(\mathrm{SO}(m,1))) = \mathcal{I}(\mathrm{SO}(m,1))$. Let

$$\mathscr{S} = \{\mathcal{I}(\mathrm{SO}(m,1)g) : N_G(\mathrm{SO}(m,1))g\Gamma \text{ is closed, } 2 \leq m \leq n-1,\ g \in G\}; \tag{12}$$

Then $\mathscr{S}$ is a countable collection of proper subspheres of $\mathbb{S}^{n-1}$ ([11, §(5.2)],[9, Cor.A]).

Theorem 1.5. *Let $\psi \in C(I,G)$ be such that $\mathcal{I} \circ \psi \in C^n(I, \mathbb{S}^{n-1})$ and for any $S \in \mathscr{S}$,*

$$(\mathcal{I} \circ \psi)^{(1)}(s) \neq 0 \quad \textit{and} \quad \mathcal{I} \circ \psi(s) \notin S \quad \textit{for Lebesgue a.e. } s \in I. \tag{13}$$

Then given a sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset C(I,G)$ such that

$$\mathcal{I} \circ \psi_k \xrightarrow{k \to \infty} \mathcal{I} \circ \psi_k, \quad \textit{in } C^n(I, \mathbb{S}^{n-1}), \tag{14}$$

and sequences $t_k \to \infty$ in $\mathbb{R}$ and $x_k \to x_0 = e\Gamma$ in G/Γ,

$$\lim_{k \to \infty} \frac{1}{|I|} \int_I f(a_{t_k} \psi_k(s) x_k)\, ds = \int_{G/\Gamma} f\, d\mu_G, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{15}$$

From this statement we will derive the following uniform version. Let $\ell_h : G \to G$ denote the left translation by $h \in G$.

Theorem 1.6. *Let $\psi : I \to G$ be a C^n-map such that if $h \in G$ and S is a proper subsphere of $\mathbb{S}^{n-1}$ then for Lebesgue a.e. $s \in I$,*

$$(\mathcal{I} \circ \ell_h \circ \psi)^{(1)}(s) \neq 0 \quad \textit{and} \quad (\mathcal{I} \circ \ell_h \circ \psi)(s) \notin S. \tag{16}$$

Then given $f \in \mathrm{C_c}(G/\Gamma)$, a compact set $\mathcal{K} \subset G/\Gamma$ and $\epsilon > 0$, there exists a neighbourhood Ω of ψ in $C^n(I,G)$ and a compact set $\mathcal{C}$ in G such that

$$\left| \frac{1}{|I|} \int_I f(g\psi_1(s)x)\, ds - \int_I f\, d\mu_G \right| < \epsilon, \quad \forall \psi_1 \in \Omega,\ x \in \mathcal{K},\ \textit{and } g \in G \setminus \mathcal{C}. \tag{17}$$

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2. Linear dynamics and growth properties of functions

Let $V = \oplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \mathfrak{g}$, and consider the $\oplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \mathrm{Ad}$ representation of G on V. We fix an inner product $\langle \cdot, \cdot \rangle$ on V and let $\|\cdot\|$ denote the associated norm.

For $\mu \in \mathbb{R}$, we define

$$V_\mu = \{v \in V : a_t v = e^{\mu t} v,\ t \in \mathbb{R}\}. \tag{18}$$

If $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\dim \mathfrak{g}}\}$ is a basis of $\mathfrak{g}$ consisting of eigen-vectors of $\{a_t\}$, then there is a basis of V_μ consisting of elements of the form $\boldsymbol{x}_{i_1} \wedge \cdots \wedge \boldsymbol{x}_{i_d}$. The eigenvalues of a_t on $\mathfrak{g}$ other than 1 are: e^t with multiplicity $n-1$, and e^{-t}

with multiplicity $n-1$. Therefore for $t>0$ the smallest eigenvalue of a_t on V is $e^{-(n-1)t}$ and the largest one is $e^{(n-1)t}$. Therefore

$$V = \oplus_{\mu=-(n-1)}^{n-1} V_\mu. \tag{19}$$

Let $q_\mu : V \to V_\mu$ be the projection associated to this decomposition.

Notation 2.1. Let $\varphi_k \to \varphi$ be a convergent sequence in $C^n(I, \mathbb{R}^{n-1})$ such that

$$\rho_0 := \inf_{s\in I} \|\varphi^{(1)}(s)\| > 0. \tag{20}$$

Let $M = Z_G(A) \cap \mathrm{SO}(n)$. Then $M \cong \mathrm{O}(n-1)$ and $Z_G(A) = AM$. We define the action of any $z \in Z_G(A)$ on $\mathbb{R}^{n-1}$ by the relation, $u(z\cdot \boldsymbol{v}) := zu(\boldsymbol{v})z^{-1}$ for all $\boldsymbol{v} \in \mathbb{R}^{n-1}$. Then M acts on $\mathbb{R}^{n-1}$ via its identification with $\mathrm{O}(n-1)$, and $a_t \cdot \boldsymbol{v} = e^t \boldsymbol{v}$.

Proposition 2.1 (Basic Lemma)**.** *Given $C>0$, there exists $R_0>0$ such that for any sequence $t_k \to \infty$ in $\mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that for any $x \in I = [a,b]$ and $v \in V$, there exists an interval $[s_k, s_k'] \subset I$ containing x such that for any $k \geq k_0$, the following conditions are satisfied:*

$$e^{t_k}(s_k' - s_k)^n < C, \tag{21}$$

$$\|a_{t_k} u(\varphi_k(s_k))v\| \geq \|v\|/R_0, \quad \textit{if } s_k > a, \tag{22}$$

$$\|a_{t_k} u(\varphi_k(s_k'))v\| \geq \|v\|/R_0, \quad \textit{if } s_k' < b. \tag{23}$$

Proof. If for every $R_0 > 0$ the above conditions are not satisfied, then after passing to a subsequence, there exist sequences $t_k \to \infty$ and $R_k \to \infty$ in $\mathbb{R}$, $v_k \to v_0$ in V with $\|v_0\| = 1$, and $[r_k, r_k'] \subset I$ such that $r_k \to r_0$, $r_k' \to r_0$ and the following holds:

$$\sup_{r_k \leq s \leq r_k'} \|a_{t_k} u(\varphi_k(s)) v_k\| \leq R_k^{-1} \tag{24}$$

$$e^{t_k}\delta_k^n \geq C, \quad \text{where } \delta_k = r_k' - r_k. \tag{25}$$

For any $k \in \mathbb{N}$, let $w_k = \varphi_k(r_k) v_k$ and

$$\varphi_{k,r_k}(s) := \varphi_k(r_k + s) - \varphi_k(r_k), \quad \forall s \in [a - r_k, b - r_k].$$

Then

$$\sup_{s\in[0,\delta_k]} \|a_{t_k} u(\varphi_{k,r_k}(s)) w_k\| \leq R_k^{-1}. \tag{26}$$

Therefore, for any $0 \leq \mu \leq n-1$,

$$\sup_{s\in[0,\delta_k]} \|q_\mu(u(\varphi_{k,r_k}(s)) w_k)\| \leq R_k^{-1} e^{-\mu t_k}. \tag{27}$$

Then by (25) we get

$$\sup_{s\in[0,\delta_k]} \|q_\mu(u(\varphi_{k,r_k}(s)) w_k)\| \leq R_k^{-1} C^{-1} \delta_k^{n\mu}. \tag{28}$$

Putting $\mu = 1$ in (28), for any $v \in V_1$ with $\|v\| = 1$,

$$\sup_{s\in[0,\delta_k]} |\langle u(\varphi_{k,r_k}(s)w_k),\, v\rangle| \leq R_k^{-1} C^{-1} \delta_k^n. \tag{29}$$

We define

$$\varphi_{0,r_0}(s) = \varphi(r_0 + s) - \varphi(s), \quad \forall s \in [a - r_0, b - r_0].$$

As $k \to \infty$, we have $R_k^{-1} \to 0$, $w_k \to w_0 = u(\varphi(r_0))v_0$, and $\delta_k \to 0$. Therefore by (29),

$$q_\mu(u(\varphi_{0,r_0}(0))w_0) = q_\mu(w_0) = 0, \quad \forall\, 0 \leq \mu \leq n-1. \tag{30}$$

To derive estimate on higher derivatives from (29) we will use the following elementary observation: If $\psi \in C^m([0,\delta], \mathbb{R})$, then there exists $\xi \in (0,\delta)$ such that

$$|\psi^{(m)}(\xi)| \leq 2^m 3^{m(m-1)/2} \delta^{-m} \sup_{s\in I} |\psi(s)|. \tag{31}$$

To prove this by induction on m, we assume that there exists $\xi_1 \in (0, \delta/3)$, and $\xi_2 \in (2\delta/3, \delta)$ such that

$$|\psi^{(m-1)}(\xi_i)| \leq 2^{m-1} 3^{(m-1)(m-2)/2} (\delta/3)^{-(m-1)} \sup_{s\in I} |\psi(s)|.$$

Then by Rolls theorem, there exists $\xi \in (\xi_1, \xi_2)$ such that

$$|\psi^{(m)}(\xi)| \leq |\psi^{(m-1)}(\xi_2) - \psi^{(m-1)}(\xi_1)| (\xi_2 - \xi_1)^{-1}.$$

Now (31) follows, because $|\xi_2 - \xi_1| \geq \delta/3$.

Combining (29) with the above observation, for each $1 \leq m \leq n$, and each k and there exists $\xi_m(k) \in (0, \delta_k)$ such that

$$\langle u(\varphi^{(m)}_{k,r_k}(\xi_m(k)))w_k,\, v\rangle \leq (2^m 3^{m(m-1)/2} C^{-1}) \delta_k^{n-m} R_k^{-1}. \tag{32}$$

There exists $a_0 < b_0$ such that $0 \in [a_0, b_0] \subset [a - r_0, b - r_0]$. Then $[a_0, b_0] \subset [a - r_k, b - r'_k]$ for all but finitely many k. As $k \to \infty$, we have $R_k \to \infty$, $\xi_m(k) \to 0$, and $\varphi_{k,r_k} \to \varphi_{0,r_0}$ in $C^n([a_0, b_0], \mathbb{R}^{n-1})$. Therefore by (30) and (32),

$$\langle u(\varphi^m_{0,r_0}(0))w_0,\, v\rangle = 0, \quad \forall v \in V_1,\ \forall\, 0 \leq m \leq n. \tag{33}$$

Hence due to Taylor's expansion,

$$\lim_{s\to 0} \|q_1(u(\varphi_{0,r_0}(s))w_0)\|/s^n = 0. \tag{34}$$

Next we will show that (34) leads to a contradiction, using finite dimensional representations of $\mathrm{SL}(2,\mathbb{R})$.

In view of (8), let $H = \mathrm{SO}(Q_2) = \mathrm{SO}(2,1) \hookrightarrow \mathrm{SO}(n,1)$. Then H is generated by $\{u(se_1)\}_{s\in\mathbb{R}}$, $A = \{a_t\}_{t\in\mathbb{R}}$ and $\{{}^{\mathfrak{t}}u(te_1)\}_{t\in\mathbb{R}}$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$. We realize H as the image of $\mathrm{SL}(2,\mathbb{R})$ under the Adjoint representation on its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ such that $\operatorname{diag}(e^t, e^{-t}) \in \mathrm{SL}(2,\mathbb{R})$ maps to $a_{2t} \in H$.

Let $\mathcal{W}$ be a finite collection of irreducible H-submodules of V such that

$$V = \oplus_{W\in\mathcal{W}} W. \tag{35}$$

For any $W \in \mathcal{W}$, let $P_W : V \to W$ denote the projection with respect to the decomposition (35). In view of Notation 2.1, for any $s \in [a - r_0, b - r_0]$, there exists $\theta(s) \in M \subset \mathrm{O}(n-1)$ such that

$$\theta(s)\cdot\varphi_{0,r_0}(s) = \|\varphi_{0,r_0}(s)\|e_1. \tag{36}$$

Let

$$\mu_0 = \max\{\mu : q_\mu(w_0) \neq 0\}. \tag{37}$$

Then $\mu_0 = \max\{\mu : q_\mu(zw_0) \neq 0\}$ for any $z \in M$. Let $W_0 \in \mathcal{W}$ be such that

$$P_{W_0}(q_{\mu_0}(\theta(0)\cdot w_0)) \neq 0. \tag{38}$$

Therefore there exist $a_1 < b_1$ such that $0 \in [a_1, b_1] \subset [a - r_0, b - r_0]$ and

$$\eta_0 := \inf_{s\in[a_1,b_1]} \|P_{W_0}(q_{\mu_0}(\theta(s)\cdot w_0))\| > 0. \tag{39}$$

By (19) and (30) we have that $-1 \geq \mu_0 \geq -(n-1)$. Recall that $\varphi_{0,r_0}(0) = 0$ and by (20), $|\varphi^{(1)}(r)| \geq \rho_0$ for all $r \in I$. Let $s \in [a_1, b_1]$ and

$$h = \|\varphi_{0,r_0}(s)\| = \|\varphi^{(1)}(r)\|s \geq \rho_0 s, \quad \text{for some } r \in [a_1, b_1]. \tag{40}$$

Then

$$\begin{aligned} \theta(s)q_1(u(\varphi_{0,r_0}(s))w_0) &= q_1(\theta(s)u(\varphi_{0,r_0}(s))w_0) \\ &= q_1(u(he_1)\theta(s)w_0). \end{aligned} \tag{41}$$

Now by the standard description of an irreducible representation of $\mathrm{SL}(2,\mathbb{R})$, we have

$$\begin{aligned} &P_{W_0}(q_1(u(he_1)\theta(s)w_0)) = q_1(u(he_1)P_{W_0}(\theta(s)w_0)) \\ &= h^{1-\mu_0}q_{\mu_0}(P_{W_0}(\theta(s)w_0)) + \sum_{\mu\leq\mu_0-1} h^{1-\mu}q_\mu(P_{W_0}(\theta(s)w_0)). \end{aligned} \tag{42}$$

Since P_{W_0} is norm decreasing and $\theta(s) \in \mathrm{O}(n-1)$, by (39) and (40), we conclude that

$$\lim_{s\to 0}\|q_1(u(\varphi_{0,r_0}(s))w_0)\|/s^{1-\mu_0} \geq \eta_0\rho_0^{1-\mu_0} > 0. \tag{43}$$

Since $0 < 1 - \mu_0 \leq n$, this contradicts (34). $\square$

Notation 2.2. For any $x \in I$, we define

$$P_{k,x}(s) = \varphi_k(x) + \varphi_k^{(1)}(x)s + \cdots + \varphi_k^{(n)}(x)s^n, \quad \forall s \in \mathbb{R}. \tag{44}$$

Corollary 2.2. *Let $R_0 > 0$ be as in Proposition 2.1 for $C = 1$. Then given a sequence $t_k \to \infty$ and $c > 0$ there exists $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$*

and $x \in I = [a,b]$ there exist $s_k, s_k' \in I$ with $x \in [s_k, s_k']$ such that for any $v \in V$, we have

$$e^{t_k}(s_k' - s_k)^n < 1 \tag{45}$$

$$\sup_{s_k \le s \le s_k'} \|a_{t_k} u(\varphi_k(s))v - a_{t_k} u(P_{k,x}(s))v\| \le c\|a_{t_k} u(\varphi(x))v\| \tag{46}$$

$$\|a_{t_k} u(\varphi_k(s_k))v\| \ge \|v\|/R_0, \quad \textit{if } s_k > a \tag{47}$$

$$\|a_{t_k} u(\varphi_k(s_k'))v\| \ge \|v\|/R_0, \quad \textit{if } s_k' < b. \tag{48}$$

Proof. Given a sequence $t_k \to \infty$, let $k_0 \in \mathbb{N}$ be as in Proposition 2.1 for the above choices of $C = 1$ and $R_0 > 0$. Let $x \in I$. By Proposition 2.1 for any $k \ge k_0$ there exists a subinterval $J_k = [s_k, s_k']$ containing x such that

$$|J_k|^n \le e^{-t_k}, \tag{49}$$

and (47) and (48) hold for all $v \in V$.

Let $\delta > 0$ be such that

$$\|u(y_1) - u(y_2)\|_V \le c, \quad \forall |y_1 - y_2| \le \delta,\ y_1, y_2 \in \mathbb{R}^{n-1}, \tag{50}$$

where $\|\cdot\|_V$ denotes the operator norm.

By (49) and equi-continuity of the family $\{\varphi_k^{(n)}\}$, there exists $k_1 \ge k_0$ such that

$$\|\varphi_k^{(n)}(x_1) - \varphi_k^{(n)}(x_2)\| \le \delta, \quad \forall x_1, x_2 \in J_k,\ \forall k \ge k_1. \tag{51}$$

Therefore by Taylor's formula,

$$|\varphi_k(s) - P_{k,x}(s)| \le \delta |J_k|^n \le \delta e^{-t_k}, \quad \forall s \in J_k,\ \forall k \ge k_1. \tag{52}$$

Let $k \ge k_1$ and $w_k = a_{t_k} u(\varphi_k(x))v$. Then due to (50) and (52), for all $s \in J_k$,

$$\begin{aligned} &\|a_{t_k} u(\varphi_k(s))v - a_{t_k} u(P_{k,x}(s))v\| \\ &\le \|u(e^{t_k}(\varphi_k(s) - \varphi_k(x)))w_k - u(e^{t_k}(P_{k,x}(s) - \varphi_k(x)))w_k\| \le \delta \|w_k\|, \end{aligned} \tag{53}$$

note that $(e^{t_k}(\varphi_k(s) - \varphi_k(x))) - (e^{t_k}(P_{k,x}(s) - \varphi_k(x))) = e^k(\varphi_k(s) - P_{k,x}(s))$. $\square$

Notation 2.3. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. Then by Notation 2.1 there exists a continuous map $z : I \to Z_G(A)$ such that

$$z(s) \cdot \varphi^{(1)}(s) = e_1,\ \forall s \in I. \quad \text{Let } R_1 := \sup_{s \in I} \|z(s)\|_V. \tag{54}$$

Proposition 2.3. *Let $R_0 > 0$ be as in Proposition 2.1 for $C = 1$. Let $\mathcal{A}$ be a linear subspace of V and $\mathcal{C}$ be a compact subset of $\mathcal{A}$. Then given $\epsilon > 0$ there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that the following holds: Given any neighbourhood Φ of $\mathcal{D}$ in V, there exist a neighbourhood Ψ of $\mathcal{C}$ and $k_2 \in \mathbb{N}$ such that and for any $v \in V$ with*

$$\|v\| \ge R_0 R_1 (\sup_{w \in \Phi} \|w\|), \tag{55}$$

a subinterval $J \subset I$, and any $k \geq k_2$ with $e^{-t_k} < |J|^n$, we have

$$\begin{aligned} &|\{s \in J : z(s)a_{t_k}u(\varphi_k(s))v \in \Psi\}| \\ &\qquad \leq \epsilon|\{s \in J : z(s)a_{t_k}u(\varphi_k(s))v \in \Phi\}|. \end{aligned} \tag{56}$$

In a special case of the above proposition when $\mathcal{A} = \{0\}$ and $\mathcal{C} = \{0\}$, we have $\mathcal{D} = \{0\}$ and the neighbourhoods Φ and Ψ can be described in terms of radii of balls centered at 0.

Proof. There exists $n_1 \in \mathbb{N}$ such that if $P : \mathbb{R} \to \mathbb{R}^{n-1}$ is a polynomial map of degree at most n and $v \in V$ then $s \mapsto u(P(s))v$ is a polynomial map of degree at most n_1.

As in [2, Prop. 4.2], there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that given an open neighbourhood Φ_1 of $\mathcal{D}$ in V there exists an open neighbourhood Ψ_1 of $\mathcal{C}$ in V contained in Φ_1 such that for any polynomial map $\zeta : \mathbb{R} \to V$ of degree at most n_1 and any bounded interval $J \subset \mathbb{R}$, if $\zeta(J) \not\subset \Phi_1$ then

$$|\{s \in J : \zeta(s) \in \Psi_1\}| \leq \epsilon|\{s \in J : \zeta(s) \in \Phi_1\}|. \tag{57}$$

Now given a bounded open neighbourhood Φ of $\mathcal{D}$ in V, we choose an open neighbourhood Φ_1 of $\mathcal{D}$ in Φ such that $\overline{\Phi_1} \subset \Phi$. Then we obtain an open neighbourhood Ψ_1 of $\mathcal{C}$ contained in Φ_1 as above. Let Ψ be an open neighbourhood of $\mathcal{C}$ contained in Ψ_1 such that $\overline{\Psi} \subset \Psi_1$.

Let $\delta > 0$ be such that 2δ-tubular neighbourhoods of $\overline{\Phi_1}$ (respectively, $\overline{\Psi}$) is contained in Φ (respectively, Ψ_1). Let $R = \sup_{w\in\Phi}\|w\|$. For $c = \delta/(R_1R) > 0$, let $k_1 \in \mathbb{N}$ be as in the Corollary 2.2. Let $k_2 \geq k_1$ be such that

$$\|z(r) - z(r')\|_V \leq \delta/R, \quad \forall |r' - r| \leq e^{-t_k},\ r, r' \in I,\ k \geq k_2. \tag{58}$$

Let $J \subset I$ be an interval and $v \in V$ with $\|v\| \geq R_0R_1R$. Let $k \geq k_2$ be such that $e^{-t_k} < |J|^n$. Define

$$E = \{s \in J : z(s)a_{t_k}u(\varphi_k(s))v \in \Psi\} \tag{59}$$

$$F = \{s \in J : z(s)a_{t_k}u(\varphi_k(s))v \in \Phi\}. \tag{60}$$

Suppose that F_1 be a connected component of F intersecting E. Let $x \in F_1 \cap E$. By Corollary 2.2 there exists $J_k = [s_k, s_k'] \subset I$ containing x such that (45) - (48) hold. Then by (47), (48), definitions of R_1, R and δ, and (58),

$$F_1 \cap E \subset \{s \in J_k \cap F_1 : z(s_k)a_{t_k}u(P_{k,x}(s))v \in \Psi_1\} \tag{61}$$

Since $e^{t_k}|J_k|^n < 1$ and $e^{t_k}|J|^n > 1$ and $x \in J \cap J_k$, we get $\{s_k, s_k'\} \cap J \smallsetminus \{a, b\} \neq \emptyset$. Therefore due to (47) and (48), $a_{t_k}u(\varphi_k(J_k \cap F_1))v \not\subset \Phi$. Hence by (46) and the choices of c and R_1,

$$z(s_k)a_{t_k}u(P_{k,x}(J_k \cap F_1))v \not\subset \Phi_1. \tag{62}$$

Therefore, since $z \mapsto \zeta(s) := z(s_k)a_{t_k}u(P_{k,x}(s))v$ is a polynomial map of degree at most n_1, by (57) applied to the interval $J_k \cap F_1$ in place of J we deduce that

$$|F_1 \cap E| \leq |\{s \in J_k \cap F_1 : z(s_k)a_{t_k}u(P_{k,x}(s))v \in \Psi_1\}| \leq \epsilon|F_1|. \tag{63}$$

Since F has at most countably many disjoint connected components intersecting E, like F_1 as above, from (63) we conclude that $|E| \leq \epsilon|F|$. $\square$

2.1. **Geometry of intersection with weakly stable subspace.**

Proposition 2.4. *Let H be a proper noncompact simple Lie subgroup of $G = \mathrm{SO}(n,1)$. Let $p_0 \in \wedge^{\dim H}\mathfrak{h} \smallsetminus \{0\}$. Then Gp_0 is closed.*

Proof. There may be a simple direct proof of this statement. Here we will quote from some earlier results.

Note that $\mathrm{Stab}(p_0) = N^1_G(H) = M_1H$, where

$$N^1_G(H) = \{g \in N_G(H) : \det(\mathrm{Ad}\, g|_{\mathrm{Lie}(H)}) = 1\}$$

and M_1 is the compact centralizer of H in G. There exists $g \in G$ such that $gHg^{-1} = \mathrm{SO}(k,1)$ and $(gM_1g^{-1})^0 = \mathrm{SO}(n-k)$ for some $2 \leq k \leq n-1$. Now $N^1_G(\mathrm{SO}(k,1))$ is a symmetric subgroup of G (see [3, pp.284–285]), and $N^1_G(\mathrm{SO}(k,1))$ stabilizes gp_0. Therefore by [4, Corollary 4.7], the orbit $Gp_0 = G(gp_0)$ is closed. $\square$

In view of (18), we define

$$V^- = \sum_{\mu<0} V_\mu, \quad V^0 = V_0, \quad V^+ = \sum_{\mu>0} V_\mu. \tag{64}$$

Then $V = V^- \oplus V^0 \oplus V^+$.

Proposition 2.5. *Let $p_0 \in V$ be as in Proposition 2.4. Let $g \in G$. Define*

$$S = S_g = \{x \in \mathbb{R}^{n-1} : u(x)gp_0 \in V^- + V^0\}. \tag{65}$$

If $S \neq \emptyset$, then either S is subsphere of a sphere in $\mathbb{R}^{n-1}$ or a proper affine subspace of $\mathbb{R}^{n-1}$.

Proof. Since the orbit Gp_0 is closed, for every $x \in S$ there exists $\xi(x) \in G$ such that

$$a_t u(x)gp_0 \xrightarrow{t\to\infty} \xi(x)p_0 \tag{66}$$

and $\xi(x)p_0 \in V^0$ is fixed by A. Let $F = N^1_G(H) = \mathrm{Stab}(p_0)$. Then the map $gF \mapsto gp_0$ from G/F to V is a homeomorphism onto Gp_0. Therefore

$$a_t u(x)gF \xrightarrow{t\to\infty} \xi(x)F, \tag{67}$$

and $A \subset \xi(x)F\xi(x)^{-1}$.

Choose any $x_0 \in S$. Let $p_1 = \xi(x_0)p_0$, $H_1 = \xi(x_0)H\xi(x_0)^{-1}$, and $F_1 = \mathrm{Stab}(p_1) = \xi(x_0)F\xi(x_0)^{-1}$. Then $A \subset F_1 = N^1_G(H_1)$. Hence $A \subset H_1$. As $\mathbb{R}$-rank of G is one, there exists a Weyl group 'element' $w \in H_1 \subset F_1$ such

that $w = w^{-1}$ and $waw^{-1} = a^{-1}$ for all $a \in A$. Now G admits a Bruhat decomposition

$$G = P^-wU^- \cup P^- = P^-U^+ \cup P^-w \tag{68}$$

(see [7, §12.14]), where

$$\begin{aligned} U^+ &= \{h \in G : a_t^{-1}ha_t \xrightarrow{t\to\infty} e\} = \{u(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^{n-1}\} \\ U^- &= \{h \in G : a_tha_t^{-1} \xrightarrow{t\to\infty} e\} = \{{}^{\mathsf{t}}u(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^{n-1}\}, \\ P^- &= \{h \in G : \overline{\{a_tha_t^{-1} : t > 0\}} \text{ is compact}\} = U^-Z_G(A) = U^-AM. \end{aligned} \tag{69}$$

Let $x \in S$. Put $g_1 = g\xi(x_0)^{-1}$, and $\xi_1(x) = \xi(x)\xi(x_0)^{-1}$. Then (67) is equivalent to

$$a_tu(x)g_1F_1 \xrightarrow{t\to\infty} \xi_1(x)F_1. \tag{70}$$

Since $w \in F_1$, by (68) there exist $b \in P^-$ and $X \in \mathfrak{u}^+$ such that

$$u(x)g_1F_1 = b\exp(X)F_1. \tag{71}$$

Now

$$a_tu(x)g_1p_1 = a_t(b\exp(X))p_1 = (a_tba_t^{-1})\exp(\operatorname{Ad}a_t(X))p_1. \tag{72}$$

Since $a_tba_t^{-1} \to b_0$ as $t \to \infty$ for some $b_0 \in Z_G(A)$, by (66)

$$\exp(e^tX)p_1 = \exp(\operatorname{Ad}a_t(X))p_1 \xrightarrow{t\to\infty} b_0^{-1}\xi_1(x)p_1. \tag{73}$$

Since U^+ is a unipotent group, the orbit U^+p_1 is a closed affine variety, and hence the map $h(U^+ \cap F_1) \mapsto hp_1$ from $U^+/(U^+ \cap F_1) \to \bar{V}$ is proper. Therefore from (73) we conclude that $\exp(X) \in F_1$. Hence by (71), $u(x)g_1F_1 = bF_1$. Therefore

$$S_g = \{x \in \mathbb{R}^{n-1} : u(x) \in P^-F_1g_1^{-1},\ A \subset F_1\}; \tag{74}$$

we have proved the inclusion "$\subset$", and the converse holds because $p_1 \in V^0$ and $P^-p_1 \subset V^0 + V^-$.

Let $\mathcal{I} : G \to P^-\backslash G \cong \mathbb{S}^{n-1}$ be the map as defined in the introduction. The right action of any $g \in G$ on $P^-\backslash G$ corresponds to a conformation transformation on $\mathbb{S}^{n-1}$. Let $\mathcal{S} : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$ be the map defined by $\mathcal{S}(x) = \mathcal{I}(u(x))$ for all $x \in \mathbb{R}^{n-1}$. Then $\mathcal{S}$ is the inverse stereographic projection. Since $A \subset F_1$, $P^- \cap F_1$ is a proper parabolic subgroup of F_1. In fact, $\mathcal{I}(F_1) = \mathcal{I}(\mathrm{SO}(k,1))$ for some $2 \leq k \leq n-1$. Hence

$$\mathcal{I}(F_1) \cong (P^- \cap F_1)\backslash F_1 \cong \mathbb{S}^{k-1}$$

is a proper subsphere of $\mathbb{S}^{n-1}$. Therefore by (74), S_g is the inverse image of a proper subsphere of $\mathbb{S}^{n-1}$ under the stereographic projection. □

Remark 2.1. In the Proposition 2.5, suppose that $\pi(N_G^1(H)) = \pi(F)$ is closed in G/Γ. Put $g = \gamma \in \Gamma$. If $x_0 \in S_\gamma$, then $g_1 = \gamma\xi(x_0)^{-1}$ and hence $F_1g_1^{-1}\Gamma = \xi(x_0)F\Gamma$ is closed. By (12) and (74), $S_\gamma = \mathcal{S}^{-1}(S)$, where $S = \mathcal{I}(F_1g_1^{-1}) \in \mathscr{S}$.

3. Limiting measure and invariance under unipotent flow

Let $\varphi_k \to \varphi$ be a convergent sequence in $C^n(I, \mathbb{R}^{n-1})$ as in Notation 2.1. Let $z : I \to Z_G(A)$ be the continuous function as in Notation 2.3 such that $z \cdot \varphi^{(1)}(s) = e_1$ for all $s \in I$. Let $g_k \to g_0$ be a convergent sequence in G. Then $x_k = g_k\Gamma \to x_0 = g_0\Gamma$ in G/Γ.

Proposition 3.1. *Given $\epsilon > 0$ there exists a compact set $\mathcal{K} \subset G/\Gamma$ such that for any sequence $t_k \to \infty$,*

$$\frac{1}{|I|}|\{s \in I : z(s)a_{t_k}u(\varphi_k(s))x_i \in \mathcal{K}\}| \geq 1 - \epsilon, \quad \textit{for all large } k \in \mathbb{N}. \tag{75}$$

Proof. Let N be a maximal unipotent subgroup of G such that $N/(N \cap \Gamma)$ is compact. Let $\mathfrak{n}$ denote the Lie algebra of N. Fix $p_N \in V \setminus \{0\}$ such that $p_N \in \wedge^{\dim \mathfrak{n}}\mathfrak{n}$. Then Γp_N is discrete (see [1]). Let

$$0 < r_3 = \inf_{\gamma \in \Gamma k \in \mathbb{N}} \|g_k \gamma p_N\|. \tag{76}$$

By Proposition 2.3 applied to $\mathcal{A} = \{0\}$, given $0 < R \leq r_3/R_0R_1$ there exists $r > 0$ such that the following holds: Given any sequence $t_k \to \infty$ there exists $k_2 \in \mathbb{N}$ such that for any interval $J \subset I$, $k \geq k_2$ with $e^{-t_k} \geq |J|^n$ and $\gamma \in \Gamma$,

$$\begin{aligned} &|\{s \in J : \|z(s)a_{t_k}u(\varphi_k(s))g_k\gamma p_N\| < r\}| \\ &\leq \epsilon \cdot |\{s \in J : \|z(s)a_{t_k}u(\varphi(s))g_k\gamma p_N\| < R\}|. \end{aligned} \tag{77}$$

By the proof of Dani's non-divergence criterion[1] for homogeneous spaces of rank one semisimple groups, the conclusion in the previous paragraph implies the existence of a compact set $\mathcal{K}$ such that (75) holds; the choice of $\mathcal{K}$ depends only on $r > 0$ chosen above, not on the sequence $\{t_k\}$. □

Take a sequence $t_k \to \infty$ in $\mathbb{R}$. Let λ_k be the probability measure on G/Γ defined by

$$\int_{G/\Gamma} f \, \mathrm{d}\lambda_k = \frac{1}{|I|} \int_I f(z(s)a_{t_k}u(\varphi_i(s))x_k) \, ds, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{78}$$

Then Proposition 3.1 implies the following:

Theorem 3.2. *After passing to a subsequence $\lambda_k \overset{k\to\infty}{\to} \lambda$ in the space of probability measures on G/Γ with respect to the weak-$*$ topology.*

Theorem 3.3. *The limit measure λ is invariant under the action of W.*

Proof. The proof follows from the same argument as in the Proof of [12, Theorem 3.1]. □

The next result says that the limit measure is null on the parabolic cylinders embedded in the cusps.

Proposition 3.4. *Let U be any maximal unipotent subgroup of G containing W and $x \in G$ such that Ux is compact. Then $\lambda(N_G(U)x) = 0$.*

Proof. (cf. [9]) Since G is a rank one group, W is contained in a unique maximal unipotent subgroup [7, §12.17]. Therefore $U = U^+$ and $N_G(U) = Z_G(A)U^+$. Let C be any compact subset of $N_G(U^+)$. Then $C_1 = \overline{\{a_{-t}Ca_t : t > 0\}}$ is compact. Given $\epsilon > 0$, let $\mathcal{K}$ be as in Proposition 3.1. Let $\mathcal{K}_1 = C_1^{-1}\mathcal{K}$.

Since Ux is compact, there exists $u \in U \setminus \{e\}$ such that $ux = x$. Then $a_{-t}ua_t \to e$ as $i \to \infty$. Therefore by [7, §1.12], $a_{-t}x \notin \mathcal{K}_1$ for all $t \geq T_0$ for some $T_0 > 0$. Therefore $a_{-T_0}Cx \cap \mathcal{K} = \emptyset$, or in other words, $Cx \cap a_{T_0}\mathcal{K} = \emptyset$.

Let $t'_k = t_k - T_0$. Then by Proposition 3.1, for all large $k \in \mathbb{N}$,

$$|\{s \in I : a_{t'_k}u(\varphi_k(s))x_k \in \mathcal{K}\}| \geq (1-\epsilon)|I|,$$

and hence $|\{s \in I : a_{t_k}u(\varphi_k(s))x_k \in a_{T_0}\mathcal{K}\}| \geq (1-\epsilon)|I|$. Since $(G/\Gamma) \setminus a_{T_0}\mathcal{K}$ is a neighbourhood of Cx, we conclude that $\lambda(Cx) \leq \epsilon$. □

4. Ratner's theorem and linearization method

Our basic goal is to prove that the following result using Ratner's description of the ergodic invariant measure for unipotent flows, and the linearization technique in combination with the linear dynamical results proved in §2.

Theorem 4.1. *Let the measure λ be as in Theorem 3.3. Suppose further that the limit function φ satisfies the following condition: For any $(n-2)$-sphere or a proper affine subspace S_1 contained in R^{n-1},*

$$|\{s \in I : \varphi(s) \in S_1\}| = 0. \tag{79}$$

Then measure λ is G-invariant.

The rest of this section is devoted to the proof of this theorem.

4.1. Positive limit measure on singular sets.

Let $\mathcal{H}$ be the collection of all closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H, and a nontrivial unipotent one-parameter subgroup of H acts ergodically on $H/H \cap \Gamma$. Then $\mathcal{H}$ is countable ([11, §5.1],[8]). For $H \in \mathcal{H}$, we define

$$N(H,W) = \{g \in G : U \subset gHg^{-1}\}, \tag{80}$$

$$S(H,W) = \bigcup_{\substack{H' \subset H,\, \dim H' < \dim H \\ H' \in \mathcal{H}}} N(H',W). \tag{81}$$

Then (see [6])

$$N(H,W) \cap N(H,W)\gamma \subset S(H,W), \quad \forall \gamma \in \Gamma \setminus N_G(H). \tag{82}$$

Suppose that λ is not G-invariant. Then by Ratner's theorem [8], since $\mathcal{H}$ is countable, there exists $H \in \mathcal{H}$ such that $\dim H < \dim G$ and

$$\lambda(\pi(N(H,W))) > 0 \quad \text{and} \quad \lambda(\pi(S(H,W))) = 0, \tag{83}$$

where $\pi : G \to G/\Gamma$ is the natural quotient map.

4.2. **Algebraic consequence of accumulation of limit measure on singular sets.** Since G is a semisimple group of real rank one and $H \in \mathcal{H}$, if H is not reductive then H is contained in a unique maximal unipotent subgroup intersecting Γ in a cocompact lattice. Hence for any $g \in N(H, W)$, we have that gHg^{-1} is contained in a maximal unipotent subgroup U of G containing W such that $U\pi(g)$ is compact, and $\pi(N(H, W)) \subset N_G(U)\pi(g)$. Now by Proposition 3.4 we have $\pi(N(H, W)) = 0$. Thus in view of (83), we conclude that H is a reductive subgroup of G.

We choose a compact set $C \subset N(H, W) \smallsetminus S(H, W)\Gamma$ and $\epsilon > 0$ such that

$$0 < \epsilon < 2\lambda(\pi(C)). \tag{84}$$

Let H^{nc} denote the subgroup of H generated by all unipotent one-parameter subgroups contained in it. Since H is a proper reductive subgroup of $G = \mathrm{SO}(n, 1)$, we have that $H^{\mathrm{nc}} \cong \mathrm{SO}(k, 1)$ for some $2 \leq k \leq n - 1$, and $H = Z_1 H^{\mathrm{nc}}$ where Z_1 is a compact central subgroup of H. Moreover $N_G(H^{\mathrm{nc}}) = M_1 H^{\mathrm{nc}}$, where M_1 is the centralizer of H^{nc} in G which is compact. Since $H \in \mathcal{H}$, we have that $\overline{H^{\mathrm{nc}}\Gamma} = H\Gamma$.

Let $\mathfrak{h}^{\mathrm{nc}}$ denote the Lie algebra associated to H^{nc} and $\ell_0 = \dim \mathfrak{h}^{\mathrm{nc}}$. Let

$$p_0 \in \wedge^{\ell_0} \mathfrak{h}^{\mathrm{nc}} \smallsetminus \{0\}. \tag{85}$$

Then $F := \mathrm{Stab}(p_0) = F$ and $N_G(H^{\mathrm{nc}})p_0 \subset \{p_0, -p_0\}$. If $\gamma \in N_G(H^{\mathrm{nc}}) \cap \Gamma$, then

$$\gamma H\Gamma = \gamma\overline{H^{\mathrm{nc}}\Gamma} = \overline{H^{\mathrm{nc}}\Gamma} = H\Gamma.$$

Therefore $\gamma \in N_G(H)$. Thus

$$\Gamma \cap N_G(H^{\mathrm{nc}}) = \Gamma \cap N_G(H). \tag{86}$$

Let $X_0 \in \mathrm{Lie}(W)$ and

$$\mathcal{A} = \{v \in \wedge^{\ell_0}\mathfrak{g} : v \wedge X_0 = 0\}. \tag{87}$$

Then $\mathcal{A}$ is a linear subspace of V. For any $g \in G$,

(88)
$$gp_0 \in \mathcal{A} \Leftrightarrow \wedge^{\ell_0}\mathfrak{h}^{\mathrm{nc}} \wedge \mathrm{Ad}(g^{-1})X_0 = 0 \Leftrightarrow H^{\mathrm{nc}} \supset g^{-1}Wg \Leftrightarrow g \in N(H, W).$$

Therefore for any $g \in G$, $g \in N(H, W) \Leftrightarrow gp_0 \in \mathcal{A}$.

Since F/H is compact, and $H\Gamma$ is closed, we have that $F\Gamma$ is closed. Therefore ΓF is closed. By Proposition 2.4, the map $gF \mapsto gp_0$ from G/F to V is proper. Therefore Γp_0 is closed in V. Hence Γp_0 is discrete.

Given any compact set $\mathcal{D}$ of $\mathcal{A}$, we define

$$\mathscr{S}(\mathcal{D}) = \{g \in G : gp_0, g\gamma p_0 \in \mathcal{D} \text{ for some } \gamma \in \Gamma \smallsetminus N_G(H^{\mathrm{nc}})\}. \tag{89}$$

Due to (82) and (86), $\mathscr{S}(\mathcal{D}) \subset S(H, W)$ and $\pi(\mathscr{S}(\mathcal{D}))$ is closed in G/Γ [6, Prop. 3.2]. Now if $\mathcal{K}$ is any compact set contained in $G \smallsetminus \pi(\mathscr{S}(\mathcal{D}))$ then there exists a neighbourhood Φ of $\mathcal{D}$ in V such that for any $g \in G$ and $\gamma_1, \gamma_2 \in \Gamma$,

(90)
$$\pi(g) \in \mathcal{K},\ \{g\gamma_1 p_0, g\gamma_2 p_0\} \subset \overline{\Phi} \Rightarrow \gamma_1 \in \gamma_2 N_G(H^{\mathrm{nc}}) \Rightarrow g\gamma_1 p_0 = \pm g\gamma_2 p_0.$$

Let $\mathcal{C} = C\cdot p_0 \cup -(C\cdot p_0) \subset \mathcal{A}$. Given $\epsilon>0$ we obtain R_0 and a compact set $\mathcal{D}\subset\mathcal{A}$ as in Proposition 2.3. Replacing $\mathcal{D}$ by $\mathcal{D}\cup -\mathcal{D}$ we assume that $\mathcal{D}$ is symmetric about 0. We choose a compact neighbourhood $\mathcal{K}$ of $\pi(C)$ contained in $G/\Gamma \smallsetminus \pi(\mathscr{S}(\mathcal{D}))$. We take any symmetric neighbourhood Φ of $\mathcal{D}$ such (90) holds. Then there exist a symmetric neighbourhood Ψ of $\mathcal{C}$ in V and $k_2\in\mathbb{N}$ such that given any subinterval J of I the following holds: If $v\in V$ with

$$\|v\| \geq R_0R_1R, \quad \text{where } R = \sup_{w\in\Phi}\|w\|, \tag{91}$$

and if $k\geq k_2$ such that $e^{-t_k} < |J|^n$, then

$$\begin{aligned} &|\{s\in J : z(s)a_{t_k}u(\varphi_k(s))v\in\Psi\}| \\ &\leq \epsilon\,|\{s\in J : z(s)a_{t_k}u(\varphi_k(s))v\in\Phi\}|. \end{aligned} \tag{92}$$

Let $\mathcal{O} = \{\pi(g) : gp_0\in\Psi, \pi(g)\in\mathcal{K}\}$. Then $\mathcal{O}$ is a neighbourhood of $\pi(C)$. Take any $k\in\mathbb{N}$. Let

$$E(k) = \{s\in I : z(s)a_{t_k}u(\varphi_k(s))\pi(g_k)\in\mathcal{O}\}. \tag{93}$$

Then by (84),

$$|E(k)| > 2\epsilon|I|, \quad \forall \text{ large } k\in\mathbb{N}. \tag{94}$$

Let B denote the ball of radius R_0R_1R centered at 0. Let

$$\Sigma_1 = \Gamma p_0 \cap B \text{ and } \Sigma_2 = \Gamma p_0 \smallsetminus B. \tag{95}$$

For $j=1,2$, let

$$E_j(k) = \{s\in E(k) : z(s)a_{t_k}u(\varphi(s))g_kv\in\Psi, \text{ for some } v\in\Sigma_j\}. \tag{96}$$

Then $E(k) = E_1(k)\cup E_2(k)$.

Now using (90) and (92), by the argument as in the proofs of [2], [6, Prop. 3.4] or [12, Prop. 4.5], there exists $k_4\in\mathbb{N}$ such that

$$|E_2(k)| \leq \epsilon|I|. \tag{97}$$

Next we want to prove that $|E_1(k)|\leq\epsilon|I|$ for all large $k\in\mathbb{N}$. Suppose this is not true. Then

$$\limsup_{k\in\mathbb{N}}|E_1(k)| \geq \epsilon|I|. \tag{98}$$

Proposition 4.2. *There exists $v\in\Sigma_1$ such that*

$$|\{s\in I : u(\varphi(s))g_0v\in V^- + V^0\}| \geq \epsilon|I|/\#(\Sigma_1). \tag{99}$$

Proof. After passing to a subsequence, there exists $v\in\Sigma_1$ such that

$$|\{s\in E(k) : z(s)a_{t_k}u(\varphi_k(s))g_kv\in\Psi\}| \geq \epsilon|I|/\#(\Sigma_1), \quad \forall \text{ large } k\in\mathbb{N}. \tag{100}$$

Let $q_+ : V\to V^+$ be the projection associated to the decomposition $V = V^-\oplus V^0\oplus V^+$ as in (64). Let

$$E^\delta = \{s\in I : q_+(u(\varphi(s))g_0v)\geq\delta\}. \tag{101}$$

Since $\varphi_k \to \varphi$ uniformly on I, there exists $k_5 \in \mathbb{N}$ such that if $k \geq k_5$ and if $s \in E^\delta$, then $q_+(z(s)u(\varphi_k(s))g_k v) \geq \delta/2$. Hence

$$z(s)a_{t_k}u(\varphi_k(s))g_k v = a_{t_k}(z(s)u(\varphi(s))g_k) \notin \Psi, \quad \forall \text{ large } k \in \mathbb{N}. \tag{102}$$

Therefore in $E^\delta \cap E_1(k) = \emptyset$ for all large $k \in \mathbb{N}$. Hence by (100) and (101),

$$|\{s \in I : u(\varphi(s))g_0 v \in V^- + V^0\}| = \lim_{\delta \to 0}|I \smallsetminus E^\delta| \geq \epsilon|I|/\#(\Sigma_1).$$

$\square$

Combining Proposition 2.5 and Proposition 4.2, there exists $\gamma \in \Gamma$ such that $\gamma p_0 = v \in \Sigma_1$ and

$$|\{s \in I : \varphi(s) \in S_{g_0\gamma}\}| \geq \epsilon|I|/\#(\Sigma). \tag{103}$$

This statement contradicts our assumption on φ as stated in (79). Therefore (98) does not hold, or in other words,

$$\limsup_{k\to\infty}|E_1(k)| < \epsilon|I|. \tag{104}$$

Therefore by (97),

$$|E(k)| \leq |E_1(k)| + |E_2(k)| < 2\epsilon|I|, \quad \forall \text{ large } k \in \mathbb{N}.$$

This contradicts (94). Thus as noted above, due to Ratner's theorem, λ is G-invariant. This completes the proof of Theorem 4.1. $\square$

Remark 4.1. If $g_0 = e$, then in (103), $S_{g_0\gamma} = S_\gamma$ for some $\gamma \in \Gamma$, and by Remark 2.1, $S_\gamma \in \mathcal{S}^{-1}(\mathscr{S})$. Therefore if $g_0 = e$, then the conclusion of Theorem 4.1 is valid if we assume the weaker condition on φ that (79) holds for all $S_1 = \mathcal{S}^{-1}(S)$, where $S \in \mathscr{S}$.

5. Deduction of the main results

The following observation allows us to deduce the results stated in the introduction from Theorem 4.1.

Proposition 5.1. *Let $\{\theta_k\}$ and $\{\psi_k\}$ be uniformly convergent sequences of continuous maps from $I \to G$ such $P^-\theta_k(s) = P^-\psi_k(s)$ for all $s \in I$. Let $\{x_k\}$ be a sequence in G/Γ and $t_k \to \infty$ be a sequence in $\mathbb{R}$. Suppose that there exists a probability measure μ on G/Γ which is $Z_G(A)$ invariant, and for any subinterval $J \subset I$ with nonempty interior and any $f \in \mathrm{C_c}(G/\Gamma)$ the following holds:*

$$\lim_{k\to\infty}\frac{1}{|J|}\int_J f(a_{t_k}\theta_k(s)x_k)\,ds = \int_{G/\Gamma} f\,d\mu. \tag{105}$$

Then for any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\lim_{k\to\infty}\frac{1}{|I|}\int_I f(a_{t_k}\psi_k(s)x_k)\,ds = \int_{G/\Gamma} f\,d\mu. \tag{106}$$

Proof. Since $P^- = U^- Z_G(A)$ (see (69)), for any $s \in I$ we express $\psi_k(s) = v(s)\zeta(s)\theta_k(s)$, where $\zeta_k(s) \in Z_G(A)$ and $v(s) \in U^-$ are such that $\{s \mapsto \zeta_k(s)\}_{k\in\mathbb{N}}$ and $\{s \mapsto v_k(s)\}_{k\in\mathbb{N}}$ are equi-continuous families of maps on I.

Let $\epsilon > 0$. Since f is uniformly continuous on G/Γ and $\overline{\{v_k(s) : s \in I, k \in \mathbb{N}\}}$ is compact in U^-, there exists $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$, $s \in I$ and $x \in G/\Gamma$, $|f(a_{t_k} v_k(s)x) - f(a_{t_k} x)| \leq \epsilon$.

Also there exists a finite partition of I onto subintervals J's such that if $s_1, s_2 \in J$ and $x \in G/\Gamma$ then $|f(\zeta_k(s_1)x) - f(\zeta_k(s_2)x)| < \epsilon$ for all $k \in \mathbb{N}$.

If we fix some $s_J \in J$, then for all $s \in J$ and $k \geq k_1$,

$$\begin{aligned}
&|f(a_{t_k}\psi_k(s)x_k) - f(\zeta(s_J)a_{t_k}\theta_k(s)x_k)| \\
&\leq |f(a_{t_k}v_k(s)\zeta_k(s)\theta_k(s)x_k) - f(a_{t_k}\zeta_k(s)\theta_k(s)x_k)| \\
&\qquad + |f(\zeta_k(s)a_{t_k}\theta_k(s)x_k) - f(\zeta_k(s_J)a_{t_k}\theta_k(s)x_k)| \leq 2\epsilon.
\end{aligned} \tag{107}$$

Since $\int f(\zeta(s_J)y)\,d\mu(y) = \int f(y)\,d\mu$, from (107) and (105),

$$\left| \int_J f(a_{t_k}\psi_k(s)x_k)\,ds - |J| \int_{G/\Gamma} f\,d\mu \right| \leq 2\epsilon|J|, \quad \forall \text{ large } k \in \mathbb{N}. \tag{108}$$

By summing this over all the J's in the partition, we deduce (106). □

Proof of Theorem 1.4. Let $\theta_k(s) = \zeta(s)u(\varphi_k(s))$ and $\psi_k(s) = u(\varphi_k(s))$ for all $s \in I$. Then by Theorem 4.1, the (105) holds for $\mu = \mu_G$. Therefore by Proposition 5.1, we have (106), which is same as (11). □

Proof of Theorem 1.5. Due to regularity of Lebesgue measure, it is enough to prove the theorem under the assumption that

$$(\mathcal{I} \circ \psi)^{(1)} \neq 0, \quad \forall s \in I. \tag{109}$$

The map $\mathcal{S} : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$ defined by $\mathcal{S}(x) = \mathcal{I}(u(x))$ is the inverse stereographic projection. Therefore without loss of generality, we may assume that there exists a sequence $\varphi_k \to \varphi$ in $C(I, \mathbb{R}^{n-1})$ such that $\mathcal{I}(\psi_k(s)) = \mathcal{I}(u(\varphi_k(s))$ and $\mathcal{I}(\psi(s)) = \mathcal{I}(u(\varphi(s))$ for all $s \in I$. Then by (109) and (13),

$$\varphi^{(1)}(s) \neq 0, \quad \text{and} \quad |\{s \in I : \varphi(s) \notin \mathcal{S}^{-1}(S)\}| = 0, \quad \forall S \in \mathscr{S}.$$

Therefore by Remark 4.1, the conclusion of Theorem 4.1 holds in the case of $x_k \to x_0 = e\Gamma$. Therefore, since $P^-\psi_k(s) = P^-u(\zeta(s)\varphi_k(s))$ for all $s \in I$, (15) follows from Proposition 5.1. □

Proof of Theorem 1.6. If the result fails to hold then there exist $f \in \mathrm{C_c}(G/\Gamma)$, $\epsilon > 0$, a sequence $x_k \to x$ in G/Γ, a sequence $\{\psi_k\}$ of functions from $I \to G$ such that $\mathcal{I} \circ \psi_k \mapsto \psi$ in $C^n(I, \mathbb{S}^{n-1})$, and an unbounded sequence $g_k \to \infty$ such that

$$\left| \frac{1}{|I|} \int_I f(g_k\psi_k(s)x_k)\,ds - \int_I f\,d\mu_G \right| \geq \epsilon. \tag{110}$$

Since $G = KA^+K$, by passing to a subsequence, for each $k \in \mathbb{N}$ we have $g_k = h'_k a_{t_k} h_k$, where $h_k \to h$ and $h'_k \to h'$ in K as $k \to \infty$, and $t_k \to \infty$ in

$\mathbb{R}$. Let $\tilde{x} \in G$ and $\tilde{x}_k \in G$ be such that $x_k = \tilde{x}_k\Gamma$, $x = \tilde{x}\Gamma$ and $\tilde{x}_k \to \tilde{x}$ as $k \to \infty$.

Let $\bar{\psi}(s) = h\psi(s)\tilde{x}$ and $\bar{\psi}_k(s) = h_k\psi_k(s)\tilde{x}_k$ for all $s \in I$ and $k \in \mathbb{N}$. Then the condition of Theorem 1.6 is satisfied for $\bar{\psi}$ in place of ψ and $\bar{\psi}_k$ in place of ψ_k; note that we have used a stronger condition on ψ that that (16) holds for all proper subspheres S of $\mathbb{S}^{n-1}$ and $h \in G$. Therefore

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f(h' a_{t_k} \bar{\psi}_k(s)\Gamma)\, ds = \int_{G/\Gamma} f(h'y)\, d\mu_G(y) = \int_{G/\Gamma} f\, d\mu_G. \tag{111}$$

Since $h'_k \to h'$ and f is uniformly continuous, this equality contradicts (110). $\square$

Proof of Theorem 1.2. As in the proof of Theorem 1.6, we need to show that given sequences $\psi_k \xrightarrow{k\to\infty} \psi$ in $C^n(I, T^1(M))$ and $t_k \xrightarrow{k\to\infty} \infty$ in $\mathbb{R}$,

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f(g_{t_k}\psi_k(s))\, ds = \int_{T^1(M)} f\, d\mu, \quad \forall f \in \mathrm{C_c}(T^1(M)). \tag{112}$$

We will deduce this statement from Theorem 1.5.

There exists a lattice Γ in $G = \mathrm{SO}(n,1)$ such that $T^1(M) \cong \mathrm{SO}(n-1)\backslash G/\Gamma$ and $T^1(\mathbb{H}^n) \cong \mathrm{SO}(n-1)\backslash G$. Moreover the geodesic flow $\{g_t\}$ on $T^1(M)$ corresponds to the translation action of $\{a_t\}$ on $\mathrm{SO}(n-1)\backslash G/\Gamma$ from the left; the action is well defined because $\mathrm{SO}(n-1) \subset Z_G(\{a_t\})$. Now the maps $\mathrm{Vis} : T^1(\mathbb{H}^n) \to \mathbb{S}^{n-1}$ and $\bar{\mathcal{I}} : \mathrm{SO}(n-1)\backslash G \to \mathbb{S}^{n-1}$ are same under the above identifications. Also the sets $\mathscr{S}$ defined in (1) and (12), as subsets of $\partial\mathbb{H}^n$ and $P^-\backslash G$ respectively, are same under the above identification.

The convergent sequence $\psi_k \to \psi$ in $C(I, T^1(M))$ can be lifted to a convergent sequence $\tilde{\psi}_k \to \tilde{\psi}$ in $C(I, T^1(\mathbb{H}^n))$. Via the above correspondence, we obtain a convergent sequence $\tilde{\tilde{\psi}}_k \to \tilde{\tilde{\psi}}$ in $C(I, G)$ such that $\mathrm{Vis}(\tilde{\psi}_k(s)) = \mathcal{I}(\tilde{\tilde{\psi}}_k(s))$ and $\mathrm{Vis}(\tilde{\psi}(s)) = \mathcal{I}(\tilde{\tilde{\psi}}(s))$. Therefore the conditions a) and b) on ψ imply the condition (13) of Theorem 1.5 for the map $\tilde{\tilde{\psi}}$. Also the required convergence property (14) is satisfied for $\{\tilde{\tilde{\psi}}_k\}$. Now any $f \in \mathrm{C_c}(T^1(M))$ can be treated as a $\mathrm{SO}(n-1)$-invariant function on G/Γ. In this case, the conclusion (15) of Theorem 1.5 holds. Therefore (112) follows. $\square$

The Theorem 1.1 is a special case of Theorem 1.2.

6. Action of $\{a_t\}$ on shrinking curves

Let $S \in \mathscr{S}$ or $S = \mathbb{S}^{n-1}$. Define

$$S^* = S \setminus \bigcup_{\substack{S' \subset S,\ \dim S' < \dim S \\ S' \in \mathscr{S}}} S'. \tag{113}$$

Let $\varphi \in C^n(I, \mathbb{R}^{d-1})$ and $g_0 \in G$. We define

$$I(S) = \{s \in I : \varphi(s) \in \mathcal{S}^{-1}(S^* g_0^{-1})\}. \tag{114}$$

By the Lebesgue density theorem, almost every $x \in I(S)$ is a *density point* of $I(S)$; that is, if I_k is any sequence of intervals in I containing x such that $|I_k| \to 0$, then $|I(S) \cap I_k|/|I_k| \to 1$ as $k \to \infty$.

Theorem 6.1. *Let $x \in I$ such that $\varphi^{(1)}(x) \neq 0$ and that x is a density point for $I(S)$, where $S = \mathbb{S}^{n-1}$. Then for any sequences $\varphi_k \to \varphi$ in $C^n(I, \mathbb{R}^{n-1})$, $g_k \to g_0$ in G, $t_k \to \infty$ in $\mathbb{R}$ and any sequence of intervals $I_k \subset [a, b]$ such that $x \in I_k$, $|I_k| \to 0$, and $|I_k|^n e^{t_k} \to \infty$ the following holds: For any $f \in \mathrm{C_c}(G/\Gamma)$,*

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f(a_{t_k} u(\varphi_k(s)) g_0)\, ds = \int_{G/\Gamma} f\, d\mu_G. \tag{115}$$

Let $\pi : G \to G/\Gamma$ denote the natural quotient map. Let $S \in \mathscr{S}$ and $m = 1 + \dim S$. Let $g \in G$ be such that $S = \mathcal{I}(\mathrm{SO}(m,1)g)$ and $N_G(\mathrm{SO}(m,1))\pi(g)$ is closed. Due to the following claim, the coset $N_G(\mathrm{SO}(m,1))g$ is uniquely defined.

We claim that if $F = \{h \in G : \mathcal{I}(\mathrm{SO}(m,1)h) = \mathcal{I}(\mathrm{SO}(m,1))\}$ then $F = N_G(\mathrm{SO}(m,1)$. To prove the claim, we note that $N_G(\mathrm{SO}(m,1)) \subset F$. In particular F is a reductive group. Since $N_G(SO(m,1))$ is a symmetric subgroup of G, by [4, Cor. 4.7], $N_G(SO(m,1))$ is a maximal reductive subgroup of G. Therefore $F = N_G(SO(m,1))$.

Let L be the subgroup of $N_G(\mathrm{SO}(m,1))$ such that $L\pi(g) = \overline{\mathrm{SO}(m,1)\pi(g)}$. Let μ_L denote the unique L-invariant probability measure on $L\pi(g)$.

Theorem 6.2. *Let $x \in I$ be such that $\varphi^{(1)}(x) \neq 0$ and x is a density point for the set $I(S)$. Then for any sequence $t_k \to \infty$ in $\mathbb{R}$, and any sequence of intervals $I_k \subset [a, b]$ such that $x \in I_k$, $|I_k| \to 0$, and $|I_k|^n e^{t_k} \to \infty$ the following holds: For any $f \in \mathrm{C_c}(G/\Gamma)$,*

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f(a_{t_k} u(\varphi(s)) g_0)\, ds = \int_{L\pi(g)} f(zy)\, d\mu_L(y), \tag{116}$$

where $z \in Z_G(A) \cap \mathrm{SO}(n)$ is such that $u(\varphi(x))g_0 \in U^- zLg$ and $u(\varphi^{(1)}(x)) \in zLz^{-1}$; z depends only on φ, x, g_0 and Lg.

Proofs of Theorem 6.1 and Theorem 6.2. Let $x_0 = \pi(g_0)$. For $k \in \mathbb{N}$, let $x_k = \pi(g_k)$ and λ_k be a probability measure on G/Γ such that for any $f \in \mathcal{C}(G/\Gamma)$,

$$\int_{G/\Gamma} f\, d\lambda_k = \frac{1}{|I_k|} \int_{I_k} f(a_{t_k} u(\varphi_k(s) x_k)\, ds.$$

First we note that Proposition 2.3 is valid for $J \subset I_k$.

Therefore the proof of Theorem 3.2 is valid in this case, and we obtain that after passing to a subsequence $\lambda_k \to \lambda$ in the space of probability measure on G/Γ.

Let $W = \{u(r\varphi^{(1)}(s)) : r \in \mathbb{R}\}$. As in Theorem 3.3, we shall show that λ is W-invariant. We will use the notation $\eta_1 \overset{\epsilon}{\approx} \eta_2$ to say that $|\eta_1 - \eta_2| \leq \epsilon$.

Let $r \in \mathbb{R}$, $\epsilon > 0$ and $f \in \mathrm{C_c}(G/\Gamma)$ be given. Due to uniform continuity of f and equi-continuity of the family $\{\varphi_k^{(1)}(s)\}$, and since $|I_k| \to 0$, for sufficiently large $k \in \mathbb{N}$ and any $s \in I_k$, the following holds:

(117)
$$\begin{aligned} f(u(r\varphi^{(1)}(x))a_{t_k}u(\varphi_k(s))x_k) &\overset{\epsilon}{\approx} f(u(r\varphi_k^{(1)}(s))a_{t_k}u(\varphi_k(s))x_k) \\ &= f(a_{t_k}u(\varphi_k(s) + e^{-t_k}r\varphi^{(1)}(s))x_k) \\ &= f(a_{t_k}u(\varphi_k(s + re^{-t_k}) + O(e^{-2t_k}))x_k) \\ &= f(u(O(e^{-t_k}))a_{t_k}u(\varphi_k(s + re^{-t_k}))x_k) \\ &\overset{\epsilon}{\approx} f(a_{t_k}u(\varphi_k(s + re^{-t_k}))x_k). \end{aligned}$$

Therefore, for sufficiently large $k \in \mathbb{N}$,

(118)
$$\begin{aligned} \int f(u(r\varphi^{(1)}(x))y)\, d\lambda_k(y) &\overset{\epsilon}{\approx} \frac{1}{|I_k|}\int_{I_k} f(a_{t_k}u(\varphi_k(s + re^{-t_k}))x_k)\, ds \\ &\overset{\epsilon}{\approx} \frac{1}{|I_k|}\int_{I_k} f(a_{t_k}u(\varphi_k(s))x_k)\, ds = \int f\, d\lambda_k, \end{aligned}$$

where the last approximation holds because
$$|I_k|^n e^{t_k} \to \infty \implies 2\frac{1}{|I_k|}|re^{-t_k}|\sup|f| \to 0.$$

From (118) we deduce that λ is W-invariant.

The proof of Proposition 3.4 goes through in this case. We can now apply Ratner's classification of ergodic invariant measures exactly as in the earlier case. Then we follow the the Proof of Theorem 4.1 for I_k in place of I. We have $E(k) = E_1(k) \cup E_2(k)$. The same proof goes through to say that for sufficiently large k, $|E_2(k)| \leq \epsilon|I_k|$. The basic difference occurs in analyzing $|E_1(k)|$. Again all the arguments are valid up to (103) for I_k in place of I; we get

$$|\{s \in I_k : \mathcal{S}(\varphi(s)) \in S'g_0^{-1}\}| \geq \epsilon|I_k|/\#(\Sigma), \tag{119}$$

for some $S' \in \mathscr{S}$. By our hypothesis, x is a density point of $I(S)$ (see (114)). Therefore in view of the definition of S^*, we deduce that $S \subset S'$.

In particular, if $S = \mathbb{S}^{n-1}$ then this is not possible. Therefore as in the Proof of Theorem 4.1 we conclude that (98) fails to hold, and in turn (94) fails to hold, and hence λ is G-invariant. Thus the proof of Theorem 6.1 is complete.

Now for Theorem 6.2 we have that $\varphi_k = \varphi$ and $g_k = g_0$ for all $k \in \mathbb{N}$. For any $s \in I(S)$, there exists $b(s) \in M$ such that $b(s) \to z$ as $s \to x$ and
$$u(\varphi(s))g_0 \in U^- b(s)Lg.$$

Therefore, since x is a density point of $I(S)$ and $\lambda_k \to \lambda$, from the definition of λ_k we conclude that $\operatorname{supp}\lambda \subset zL\pi(g)$. Since $\lambda(\pi(N(H,W)) > 0$, we conclude that $S'g_0^{-1} \subset Sg_0^{-1}$. Thus $S = S'$ and $H^{\mathrm{nc}} \cong \mathrm{SO}(m,1)$, and $\operatorname{supp}(\lambda) \subset$

$g'N_G(H^{\mathrm{nc}})\pi(e)$ for some $g' \in G$ such that $AW \subset g'N_G(H^{\mathrm{nc}})(g')^{-1}$. Since $\lambda(\pi(S(H,W))) = 0$, we deduce that each W-ergodic component of λ is invariant under $g'H^{\mathrm{nc}}(g')^{-1}$. Therefore, since

$$\mathrm{supp}(\lambda) \subset zL\pi(g) \quad \text{and} \quad L\pi(g) = \overline{\mathrm{SO}(m,1)\pi(g)},$$

by dimension consideration, we conclude that $\mathrm{supp}(\lambda) = zL\pi(g)$ and $\lambda = z\mu_L$. This completes the proof of Theorem 6.2. $\square$

7. Evolution of shrinking curves under geodesic flow

Let the notation be as in §1.1. As a consequence of Theorem 6.1 we obtain the following.

Theorem 7.1. *Let $\psi \in C^n(I, T^1(M))$. Let $x \in I$ be such that $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(x) \neq 0$ and that x is a density point of $I(M)$. Then for any sequence $\psi_k \to \psi$ in $C^n(I, T^1(M))$, a sequence $t_k \to \infty$, and a sequence I_k of subintervals of I containing x such that $|I_k| \to 0$ and $|I_k|^n e^{t_k} \to \infty$ the following holds:*

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f(g_{t_k}\psi_k(s))\, ds = \int_{T^1(M)} f\, d\mu_M, \quad \forall f \in \mathrm{C_c}(T^1(M)). \tag{120}$$

As a consequence of Theorem 6.2 we deduce the following:

Theorem 7.2. *Let $\psi \in C^n(I, T^1(M))$. Let $M_1 \in \bar{\mathscr{S}}$ and $S \in \mathscr{S}(M_1)$. Let $x \in I$ be such that $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(x) \neq 0$ and x is a density point of $I(S)$. Then given any sequence $t_k \to \infty$ in $\mathbb{R}$ and a sequence of subintervals I_k of I containing x such that $|I_k| \to 0$ and $|I_k|^n e^{t_k} \to \infty$ the following holds:*

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f(a_{t_k}\psi(s))\, ds = \int_{T^1(M_1)} f\, d\mu_{M_1}, \quad \forall f \in \mathrm{C_c}(T^1(M)). \tag{121}$$

The conclusion of Theorem 1.3 can be deduced from Theorem 7.2 using using regularity Lebesgue measure and standard arguments of measure theory.

7.1. Geodesic evolution of faster shrinking curve.

We can also obtain following variations of Theorem 7.1 and Theorem 7.2.

Theorem 7.3. *In the statement of Theorem 7.1 suppose that $\psi_k \to \psi$ in $C^{2n-2}(I, T^1(M))$. Then given a sequence $t_k \to \infty$, and a sequence of subintervals I_k of I containing x such that $|I_k| \to 0$ and $|I_k|^2 e^{t_k} \to \infty$, the equation* (120) *holds.*

Theorem 7.4. *In the statement of Theorem 7.2 suppose that $\psi \in C^{2n-2}(I, T^1(M))$. Then given a sequence $t_k \to \infty$, and a sequence of subintervals I_k of I containing x such that $|I_k| \to 0$ and $|I_k|^2 e^{t_k} \to \infty$, the equation* (121) *holds.*

It is interesting to compare these statements with the results in [13].

To prove the above theorems using the method of this article, the only property required to be verified is the following variation of Proposition 2.1.

Proposition 7.5 (Basic Lemma-II). *Let $\varphi_k \to \varphi$ in $C^{2n-2}(I, \mathbb{R}^{d-1})$. Given $C > 0$, there exists $R_0 > 0$ such that for any sequence $t_k \to \infty$ in $\mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that for any $x \in I = [a, b]$ and $v \in V$, there exists an interval $[s_k, s'_k] \subset I$ containing x such that for any $k \geq k_0$, the following conditions are satisfied:*

$$e^{t_k}(s'_k - s_k)^2 < C, \tag{122}$$

$$\|a_{t_k} u(\varphi_k(s_k))v\| \geq \|v\|/R_0, \quad \text{if } s_k > a, \tag{123}$$

$$\|a_{t_k} u(\varphi_k(s'_k))v\| \geq \|v\|/R_0, \quad \text{if } s'_k < b. \tag{124}$$

Proof. We follow the strategy of the proof of Proposition 2.1. We will now highlight some crucial modification required in the proof.

First (25) is replaced by $e^{t_k}\delta_k^2 \geq C$. We put $\mu = n - 1$ in (27) to get

$$\sup_{s \in [0, \delta_k]} \|q_{n-1}(u(\varphi_{k,r_k}(s))w_k)\| \leq R_k^{-1} C^{-1} \delta_k^{2(n-1)}.$$

Therefore in place of (34) we will have

$$\lim_{s \to 0} \|q_{n-1}(u(\varphi_{0,r_0}(s))w_0)\|/s^{2n-2} = 0. \tag{125}$$

Now following the further arguments using the $\mathrm{SL}(2, \mathbb{R})$-representation theory, we will obtain an analogue of (42) for q_{n-1} involving $h^{(n-1)-\mu_0}$ in the highest order term. Therefore (43) will become

$$\lim_{s \to 0} \|q_{n-1}(u(\varphi_{0,r_0}(s))w_0)\|/s^{(n-1)-\mu_0} \geq \eta_0 \rho_0^{n-1-\mu_0} > 0.$$

Since $n - 1 - \mu_0 \leq 2n - 2$, this will contradict (125). $\square$

Remark 7.1. Using Proposition 7.5, we can obtain an analogue of Corollary 2.2 for $P_{k,x}(s) = \varphi_k(x) + s\varphi_k^{(1)}(s)$. Thus for linearization technique, we can approximate a $C^{2(n-1)}$-curve φ_k at any $s \in I$ by its tangent line, rather than a polynomial curve.

References

[1] S. G. Dani. On orbits of unipotent flows on homogeneous spaces. *Ergodic Theory Dynamical Systems*, 4(1):25–34, 1984.

[2] S. G. Dani and G. A. Margulis. Limit distributions of orbits of unipotent flows and values of quadratic forms. In *I. M. Gelfand Seminar*, pp. 91–137. Amer. Math. Soc., 1993.

[3] Alex Eskin, Shahar Mozes and Nimish Shah. Unipotent flows and counting lattice points on homogeneous varieties. *Ann. of Math. (2)* 143(2):253–299, 1996.

[4] A. Gorodnik, H. Oh and N. Shah *Integral points on symmetric varieties and Satake compactifications*, *Amer. J. Math.* (to appear). `http://arxiv.org/abs/math/0610497`.

[5] D. Y. Kleinbock and G. A. Margulis. Flows on homogeneous spaces and Diophantine approximation on manifolds. *Ann. of Math. (2)*, 148(1):339–360, 1998.

[6] Shahar Mozes and Nimish Shah. On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems*, 15(1):149–159, 1995.

[7] M.S. Raghunathan. *Discrete subgroups of Lie groups.* Springer-Verlag, 1972.

[8] Marina Ratner. On Raghunathan's measure conjecture. *Ann. of Math. (2)*, 134(3):545–607, 1991.
[9] Marina Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991.
[10] Nimish A. Shah. Closures of totally geodesic immersions in manifolds of constant negative curvature. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 718–732. World Sci. Publishing, River Edge, NJ, 1991.
[11] Nimish A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289(2):315–334, 1991.
[12] Nimish A. Shah. Limiting distributions of curves under geodesic flow on hyperbolic manifold. 25 pages. `http://arxiv.org/abs/0708.4093`
[13] Andreas Strömbergsson. On the uniform equidistribution of long closed horocycles. *Duke Math. J.*, 123(3):507–547, 2004.

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